Journal of Novel Applied Sciences
Available online at www.jnasci.org
©2014 JNAS Journal-2014-3-2/178-180
ISSN 2322-5149 ©2014 JNAS

Finite Prüfer rank of G and its Product

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ABSTRACT: A group G has finite Prüfer rank r=r(G) if every finitly generated subgroup of G can be generated by at most r elements, and r is the least positive integer with this properly. In this paper we show that if the locally soluble group G=AB with finite Prüfer rank is the product of two subgroups A and B, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of A and B.

Keywords: finite Prüfer rank, locally soluble group, product .

INTRODUCTION

In 1968 N.F. Sesekin (see [19]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972. Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition χ, when does G have the same finiteness condition χ ?(see 20)

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see [1], [2],[3],[4] and [6]) , N.S. Chernikov (see [5]), S. Franciosi, F. de Giovanni (see [3],[6]), O.H.Kegel (see [8]), J.C.Lennox (see [12]) , D.J.S. Robinson(see [9] and [12]), J.E. Roseblade(see [13]), Y.P.Sysak(see [19] and[20]), J.S. Wilson(see [23]), and D.I.Zaitsev(see [11] and [18]).

Now, in this paper, we study the finite Prüfer rank of locally soluble group G and its relations, and the end we prove that if the locally soluble group G=AB with finite Prüfer rank is the product of two subgroups A and B, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of A and B.

2. Priliminaries : (elementary properties and theorems.)

2.1. Lemma: Let the finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π - *group*, for a set π of primes, then there exist Hall π-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π-subgroups of G.

Proof: Let A_1, B_1, and G_1 be Hall π-subgroups of A, B, and G, respectively. Since G is a D_π - *group*, there exist elements x and y such that A_1^x *and* B_1^y are both contained in G_1. It follows from Lemma 2.4 that $A^x = A^z$ and $B^y = B^z$ for some z in G. Thus $A_0 = A_1^{xz^{-1}}$ and $B_0 = B_1^{yz^{-1}}$ are Hall π-subgroups of A and B, respectively, which are both contained in $G_0 = G_1^{yz^{-1}}$. Clearly the order of $A_0 \cap B_0$ is bounded by the maximum π-divisor n of the order of $A \cap B$ since $|G| = \frac{|A|.|B|}{|A \cap B|}$, It follows that $|G_0| = \frac{|A_0|.|B_0|}{n} \leq \frac{|A_0|.|B_0|}{|A_0 \cap B_0|} = |A_0B_0|$. Therefore $A_0B_0=G_0$ is a Hall π-subgroup of G.

2.2.Corollary: Let the finite group G=AB be the product of two subgroups A and B .Then for each prime p there exist Sylow p-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Sylow p-subgroup of G.
Proof: See [5].

2.3.Lemma: (See [13]) If N is a maximal abelian normal dubgroup of a finite p-group G, then $r(G) \leq \frac{1}{2} r(N)(5\,r(N)+1).$
Proof : Since $C_G(N)=N,$ the factor group G/N is isomorphic with a p-group of automorphism of N. Thus G/N has perufer rank at most $\frac{1}{2} r(N)(5r(N)-1)$ (See [15], part 2,lemma 7.44), and hence $r(G) \leq \frac{1}{2} r(N)(5r(N)+1)$.

2.4.Lemma : Clearly subgroups and homomorphic images of groups with finite Prüfer rank also have finite Prüfer rank.

Proof : See [5].
2.5. Main Theorem: If the locally soluble group G=AB with finite Prüfer rank is the product of two subgroups A and B, then the Prüfer rank of G is bounded by a function of the Prüfer ranks of A and B.

Proof : First, let G be a finite p-group for some prime p. If N is a maximal abelian normal subgroup of G, by Lemma 2.3 we have $r(G) \leq \frac{1}{2} r(N)(5r(N)+1).$ Hence it is enough to prove that r=r(N) is bounded by a function of the maximum s of r(A) and r(B). The socle S of N is an elementary abelian group of order p^r. Clearly it is sufficient to prove the theorem for the factorizer X(S) of S. Therefore we may suppose that the group G has a triple factorization G=AB=AK=BK, where K is an elementary abelian normal subgroup of G of order p^r.

Let e be the least positive integer such that A^{p^e} is contained in B. By Lemma 4.3.3 of [4], we have $|A:A\bigcap B| \leq |A:A^{p^e}| \leq p^{eg(s)-s^2}$ Where $g(s)=\frac{1}{2}s(3s+1).$ *Since*

$$|G|=\frac{|A|.|B|}{|A\bigcap B|}=\frac{|B|.|K|}{|B\bigcap K|},$$

It follows that $|K|=|A:A\bigcap B|.|B\bigcap K| \leq p^{eg(s)-s^2}p^s=p^{eg(s)-s^2+s}$. Hence $r \leq eg(s)-s^2+s \leq eg(s).$ Therefore it is enough to show that $e \leq g(s)+3.$ Therefore it is enough to show that $e \leq g(s)+3.$

Clearly we may suppose that e>1. Let a be an element of A such that $a^{p^{e-1}}$ is not in B, and write $a^{p^{e-1}}$ =xb, with x in K and b in B. Then $[x,a^{p^{e-2}}] \neq 1,$ because otherwise

$$b^p=(x^{-1}a^{p^{e-2}})^p=x^{-p}a^{p^{e-1}}=a^{p^{e-1}},$$

contrary to the choice of a. As K has exponent p, it follows from the usual commutator laws that .

$$[x,a^{p^{e-2}}]=\prod_{i=1}^{p^{e-2}}[x,{}_ia]^{(p^{e_i-2})}=[x,p^e.2a].$$

Thus

$$[K,\underbrace{G,\ldots G}_{\leftarrow p^{e-2}\rightarrow}]\neq 1,$$

and so |K|> $p^{p^{e-2}}$ since G is a finite p-group. Therefore $p^{p-2}<r \leq eg(s).$ If $e \geq g(s)+4$, then $p^{e-2} \geq 2^{e-2} > (e+1)(e-4) \geq (e+1)g(s) > eg(s).$

This contradiction shows that $e \leq g(s)+3.$

Suppose now that G=AB is an arbitrary finite soluble group. For each prime p, by Corollary 2.2 there exist Sylow p-subgroups A_p of A and B_p of B such that $G_p=A_pB_p$ is a Sylow p-subgroup of G. As was shown above, $r(G_p)$ is bounded by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by a function f(s) of the maximum s of r(A) and r(B), and this does not depend on p. Thus every subgroup of prime-power order of G can be generated by at most f(s) elements. Application of Theorem 4.2.1 of [4] yields that every subgroup of G can be generated by at most f(s)+1 elements, and hence the Prüfer rank of G is bounded by f(s)+1. This proves the theorem is the finite case.

Let G=AB be an arbitrary locally soluble group with finite Prüfer rank. If N is a finite normal subgroup of G, and X=X(N) is its factorizer, then the index $|X : A\cap B|$ *is finiteby Lemma 1.1.5.Let Y be the core of* $A\cap B$ *in* X. Since the factorized group X/Y is finite, it follows from the first part of the proof that thePrüfer rank of X/Y is bounded by a function of the Prüfer ranks of A and B. As $r(N)\le r(X)\le r(Y)+r(X/Y)\le r(A)+r(X/Y)$(e.g.see Robinson 1972, Part 1,Lemma 1.44) we obtain that there exists a function h such that $r(N)\le h(r(A),r(B))=k,$ for every finite normal subgroup N of G. Clearly the same holds for every finite normal section of G.

Let T be the maximum periodic normal subgroup of G. If p is a prime, the group $\overline{T}=T/O_{p'}(T)$ is Chernikov by Lemma 3.2.5 of [4] (See also [16]). Let $\overline{J}$ be the finite residual of $\overline{T}$, and $\overline{S}$ the socle of $\overline{J}$. Since $\overline{S}$ and $\overline{T}/\overline{J}$ are finite, it follows that $r(\overline{T})\le r(\overline{J})+r(\overline{T}/\overline{J})=r(\overline{S})+r(\overline{T}/\overline{J})\le 2k.$

As the Sylow p-subgroups of T can be embedded in $\overline{T}$, they have Prüfer rank at most 2k. Application of Theorem 4.2.1 of [4] (See also [14]). yields that every finite subgroup of T can be generated by atmost 2k+1 elements. Hence r(T)$\le 2k+1.$

The group G/T is soluble (Robinson 1972, Part 2, Lemma 10.39), and so the setoff primes $\pi(G/T)$ is finite by Lemma 4.1.5 of [5](See also [15]). It follows from Lemma 4.1.4 of [4] (See also [15]) that there exists in G a normal series of finite length $T\le G_1\le G_2\le G,$ where G_1/T is torsion-free nilpotent, G_2/G_1 is torsion-free abelian, and G/G_2 is finite. Therefore

$$\begin{aligned} r(G) &\le r(T)+r(G_1/T)+r(G_2/G_1)+r(G/G_2) \\ &\le r(T)+r_0(G)+r(G/G_2) \\ &\le r_0(G)+3k+1. \end{aligned}$$

By theorem 4.1.8 of [4] (See also [3]) we have that $r_0(G)\le r_0(A)+r_0(B).$

Moreover, $r_0(A)\le r(A)$ *and* $r_0(B)\le r(B)$ *by Lemma 4.3.4* of [4] (See also [9]). Therefore $r(G)\le r(A)+r(B)+3k+1.$ The theorem is proved.

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